Indicative Conditionals and Probabilities

Marco Degano

Philosophical Logic 2024
5 December 2024

Readings

Optional:

- E. Adams, *The Logic of Conditionals* (1975)

Plan

1. Indicative Conditionals
2. Probability and Logic
3. Adams Account
4. Lewis Triviality

Outline

Indicative and Subjective

Recall: we distinguish between indicative conditionals and subjective conditionals (counterfactuals).

(1) a. If Oswald hadn't shot Kennedy, then someone else would have.

b. $\phi \rightsquigarrow \psi$

(2) a. If Oswald didn't shoot Kennedy, then someone else did.

b. $\phi \rightarrow \psi$

Indicative Conditionals and Material Implication

$\phi \to \psi \equiv \phi \supset \psi \equiv \neg\phi \vee \psi \equiv \neg(\phi \wedge \neg\psi)$?

We seem to accept the following inferences:

Or-to-if: $\phi \vee \psi \models \neg\phi \to \psi$

Not-and-to-if: $\neg(\phi \wedge \psi) \models \phi \to \neg\psi$

Moreover, indicative conditionals must entail material conditional to make modus ponens valid.

$\to$-to-$\supset$: $\phi \to \psi \models \phi \supset \psi$

Should the material conditional and the indicative conditional be equivalent?

For Indicative as Material

Ad absurdum conditionals are good:

(3) If you can run 100 km without stopping, I will eat my hat.

We want to make the claim that the antecedent doesn't hold because the consequence is absurd.

Gibbard's Collapse Theorem: if $\phi \to \psi$ is stronger than $\phi \supset \psi$, and we accept *Conditional Proof*, *Import-Export* and the *Deduction Theorem*, then $\to$ is equivalent to $\supset$

Gibbard's Collapse Theorem

$(\mathrm{P1})$ $\phi \to \psi \models \phi \supset \psi$ (assumption)

$(\mathrm{P2})$ If $\phi \models \psi,$ then $\models \phi \to \psi$ (Conditional Proof)

$(\mathrm{P3})$ $\phi \to (\psi \to \chi) \equiv (\phi \wedge \psi) \to \chi$ (Import-Export)

(1) $(\phi \supset \psi) \to (\phi \to \psi) \equiv ((\phi \supset \psi) \wedge \phi) \to \psi$ (Instance of (P3))

(2) $((\phi \supset \psi) \wedge \phi) \to \psi$ (by $(\phi \supset \psi) \wedge \phi \models \psi$ and (P2))

(3) $(\phi \supset \psi) \to (\phi \to \psi)$ (by (1) and (2))

(4) $(\phi \supset \psi) \to (\phi \to \psi) \models (\phi \supset \psi) \supset (\phi \to \psi)$ (by (P1))

(5) $(\phi \supset \psi) \supset (\phi \to \psi)$ (by (3) and (4))

(6) $\phi \supset \psi \models \phi \to \psi$ (by Deduction Theorem)

Against Indicative as Material

Paradoxes of Material Implication.

Consider q = 'It rains' and p = 'The proof is wrong'.

$q \models p \supset q$

$\neg p \models p \supset q$

Strict Implication

Assuming a strict implication analysis also does not work.

$\Box(\phi \to \psi) = \forall w \in W :$ if $w \models \phi,$ then $w \models \psi$

Still, we generate paradoxes of strict implication:

$\Box q \models \Box(p \supset q)$

$\Box\neg p \models \Box(p \supset q)$

Outline

Probabilities

A probability function P is a function from $\mathcal{L}$ to R s.t.

1. $P(\phi) \geq 0$ for all $\phi \in \mathcal{L}$
2. If $\models \phi$, then $P(\phi) = 1$
3. If $\models \neg(\phi \wedge \psi)$ [ϕ and ψ are mutually exclusive], then $P(\phi \vee \psi) = P(\phi) + P(\psi)$

Note that then for any ϕ, $P(\phi) \in [0, 1]$. It also follows that logically equivalent formulas have the same probability.

Some facts

$$P(\neg\phi) = 1 - P(\phi)$$

By (2), $P(\phi \vee \neg\phi) = 1$. By (3), $P(\phi \vee \neg\phi) = P(\phi) + P(\neg\phi)$. Thus, $1 - P(\phi) = P(\neg\phi)$

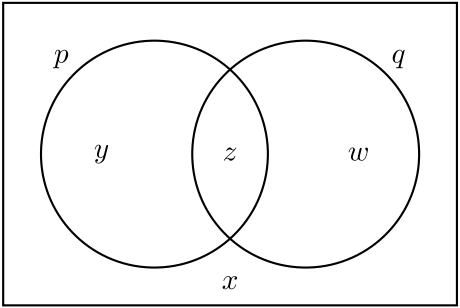
$$P(\phi \wedge \psi) \leq P(\phi)$$

$P(\phi) = P((\phi \wedge \psi) \vee (\phi \wedge \neg\psi)) = P(\phi \wedge \psi) + P(\phi \wedge \neg\psi)$. Thus $P(\phi \wedge \psi) = P(\phi) - P(\phi \wedge \neg\psi)$

Exercise: Show that $P(\phi \vee \psi) = P(\phi) + P(\psi) - P(\phi \wedge \psi)$

Venn Diagram

It is often convenient to represent probabilities by means of Venn Diagrams



$$P(p) = y + z$$

$$P(p \wedge q) = z$$

$$P(p \vee q) = y + z + w$$

True Premises

$\phi_1, \phi_2 \models_{CL} \psi$

If the premises have probability 1, then the conclusion should have probability 1

if $P(\phi_1) = P(\phi_2)$ = 1, then $P(\psi) = 1$ (for all P)

But what if premises do not have probability 1?

Probabilities can decrease

$p \vee q, p \supset q \models_{CL} q$

In this case, we can derive that $P(q) = P(p \vee q) + P(p \supset q) - 1$.

So, if we set the probability of the premises, we also determine the probability of the conclusion.

However, in general, we cannot derive the closed form of the conclusion. Lower and higher bounds is typically what we can prove.

$p \supset q, p \models_{CL} q$

If $P(p \supset q) \geq a$, $P(p) \geq b$, then $P(q) \geq a + b - 1$

In particular, if $P(p \supset q) \geq 1 - \epsilon$ and $P(p) \geq 1 - \epsilon$ (if both premises are almost true), then $P(q) \geq 1 - 2\epsilon$

Probabilistic Entailment

$\Gamma \models^1 \phi$ iff $\forall P$: if $\forall \gamma \in \Gamma : P(\gamma) = 1$, then $P(\phi) = 1$

$\Gamma \models^2 \phi$ iff $\forall P$: if $\forall \gamma \in \Gamma : P(\gamma) \geq n$, then $P(\phi) \geq n$

But $\models^2$ does not support MP.[1]

[1] It also gives rise to the Lottery Paradox, when knowledge operators are considered. For the failure of modus ponens, set $n = 0.3$. Say that $P(p) = 0.6$, $P(q) = 0.21$, $P(p \wedge q) = 0.2$. Then if we take the material implication, $P(p \supset q) = 0.6$. As we will later see, even if the take the conditional probability $P(q|p) = \frac{1}{3}$, MP does not work here.

Probabilistic Entailment

$U_P(\phi) = P(\neg\phi) = 1 - P(\phi)$ U(ncertainty)

Probabilistic Entailment:

$\Gamma \models_P \phi$ iff $\forall P : U_P(\phi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$

Read: the uncertainty of the conclusion is no greater than the uncertainty of the premise.

While other equivalent definitions are possible (next slide), we will adopt the former here and in the exercises. However, a useful fact is the following.

Equivalently (Adams, Theorem 3):

$\gamma_1, \ldots, \gamma_n \models_P \phi$ iff $\forall P : \forall \gamma_i : P(\gamma_i) \geq 1 - \epsilon$, then $P(\phi) \geq 1 - n\epsilon$.

Suppes and Adams

$\Gamma \models_s \phi$ iff for all P: $U_P(\phi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$ Patrick Suppes

Suppes is what we defined before $\models_P$.

$\Gamma \models_a \phi$ iff for all $\epsilon \geq 0$, there is a $\delta \geq 0$ s.t. for all P: if $P(\neg\gamma) \leq \delta$, then $P(\neg\phi) \leq \epsilon$. Ernest Adams

Read: whenever your premises are mostly correct within a tolerance δ, your conclusion will also be mostly correct within a tolerance ϵ.

We can prove that (Adams, Theorem 3)

$$(a)\ \Gamma \models_{CL} \phi \text{ iff } (b)\ \Gamma \models_s \phi \text{ iff } (c)\ \Gamma \models_a \phi$$

From Classical to Suppes

$\Gamma \models_s \phi$ iff for all P: $U_P(\phi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$

From (a) to (b)

Suppose that $\Gamma \not\models_s \phi$. So there is a P s.t. $1 - P(\phi) \geq \sum_{\gamma \in \Gamma} P(\neg\gamma)$. Reordering, $1 \geq \sum_{\gamma \in \Gamma} P(\neg\gamma) + P(\phi)$.

Note that $P(\neg a \vee b) = P(\neg a) + P(b) - P(\neg a \wedge b)$.

Thus $1 \geq \sum_{\gamma \in \Gamma} P(\neg\gamma) + P(\phi) \geq P(\gamma_1 \wedge \ldots \wedge \gamma_n \supset \phi)$ for any subset $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$.

But all tautologies have probability 1 and thus no such implication $\gamma_1 \wedge \ldots \wedge \gamma_n \supset \phi$ is a tautology. Hence $\Gamma \not\models_{CL} \phi$.

From Suppes to Adams

$\Gamma \models_s \phi$ iff for all P: $U_P(\phi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$

$\Gamma \models_a \phi$ iff for all $\epsilon \geq 0$, there is a $\delta \geq 0$ s.t. for all P: if $P(\neg\gamma) \leq \delta$, then $P(\neg\phi) \leq \epsilon$.

From (b) to (c)

Suppose that $\Gamma \models_s \phi$ and $\epsilon \geq 0$. If Γ is finite, pick $\delta = \frac{\epsilon}{|\Gamma|}$.

Then $P(\neg\phi) \leq \sum_{\gamma \in \Gamma} P(\neg\gamma) \leq |\Gamma| \cdot \delta = \epsilon$.

Hence $P(\neg\phi) \leq \epsilon$. Thus $\Gamma \models_a \phi$.

From Adams to Classical

$\Gamma \models_a \phi$ iff for all $\epsilon \geq 0$, there is a $\delta \geq 0$ s.t. for all P: if $P(\neg\gamma) \leq \delta$, then $P(\neg\phi) \leq \epsilon$.

From (c) to (a)

Suppose $\Gamma \not\models_{CL} \phi$. For any $\epsilon \geq 0$, we can simply take a valuation witnessing $\Gamma \not\models_{CL} \phi$, which is itself a probability function P making $P(\phi) = 0$ and $P(\gamma) = 1$, for all $\gamma \in \Gamma$. Hence $\Gamma \not\models_a \phi$.

(A similar proof works to show from (b) to (a)).

Material Conditional and Probability

Should we take $P(\phi \supset \psi)$ as the meaning of the indicative conditionals?

We have already discussed why the material conditional is not adequate with respect to the validity of certain inferences.

From the perspective of probabilities, consider the following.

Given that a card has been picked at random from a standard 52 card deck, to what degree would you believe: *The card is a king, IF it is red.*

The answer is $\frac{1}{13}$.

But $P(r \supset k) = P(\neg r \vee k) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} > \frac{1}{13}$

Outline

1. Indicative Conditionals

2. Probability and Logic

3. Adams Account

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Conditional Probability

Conditional probability is the probability of an event happening given that another event has already occurred.

Imagine you are holding a deck of cards. What is the probability of drawing a heart? $\frac{13}{52}$

Now suppose that I tell you that the card you are drawing is red. What is the probability that the card is a heart, then?

You now know you are only dealing with 26 red cards, half of which are hearts. So the probability of drawing a heart, given that the card is red, is $\frac{13}{26} = 0.5$.

In general the probability of ψ, given ϕ is

$P(\psi|\phi) = \frac{P(\psi \wedge \phi)}{P(\phi)}$

Note that $P(\phi \wedge \psi) = P(\psi|\phi) \cdot P(\phi)$

Chain rule: $P(\phi \wedge \psi \wedge \chi) = P(\phi) \cdot P(\psi|\phi) \cdot P(\chi|(\phi \wedge \psi))$.

Adams' thesis

Asserting an indicative conditional $\phi \to \psi$ amounts to $P(\psi|\phi)$
This gives the correct prediction for the case examined before

(4) a. The selected card is a king, if it is red.
b. $P(k|r) = \frac{1}{13}$

Extending the language

Recall: $\Gamma \models_P \phi$ iff $\forall P : U_P(\phi) \leq \sum_{\gamma \in \Gamma} U_P(\gamma)$

Assume now that we extend the language with a new operator $\rightarrow$ s.t. $P(p \rightarrow q) = P(q|p)$.

Note that if $P(\phi \rightarrow \psi) \geq 1 - \epsilon$ and $P(\phi) \geq 1 - \epsilon$, then $P(\psi) \geq 1 - 2\epsilon$

$P(\psi \wedge \phi) = P(\psi|\phi) \cdot P(\phi) = (1 - \epsilon)^2$

Note that $P(\psi \wedge \phi) \leq P(\psi)$. Hence $P(\psi) \geq (1 - \epsilon)^2 = 1 + \epsilon^2 - 2\epsilon \geq 1 - 2\epsilon$

What does this tell us? Modus ponens is valid.

Paradox of Material Implication

$\phi_1, \ldots, \phi_n \models_P \psi$ iff for all P:
$U_P(\phi_1) + \cdots + U_P(\phi_n) \geq U_P(\psi)$

With one premise $\phi \models_P \psi$ iff for all P, $U_P(\phi) \geq U_P(\psi)$
iff for all P, $P(\phi) \leq P(\psi)$

$q \not\models_P p \to q$. Take a probability assignment P s.t. $P(q) = 0.9$ and $P(p \to q) = 0.1$.

The paradoxes of the material implication do not arise under this treatment.

Finding invalidites

To show that $\phi_1, \ldots, \phi_n \not\models_P \psi$

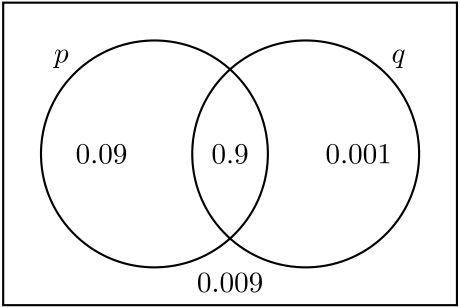
you need to find a probability function P s.t.
$U_P(\phi_1) + \cdots + U_P(\phi_n) < U_P(\psi)$

It is often useful to draw Venn Diagrams with the values of the respective probabilities.

Some Invalid Inferences

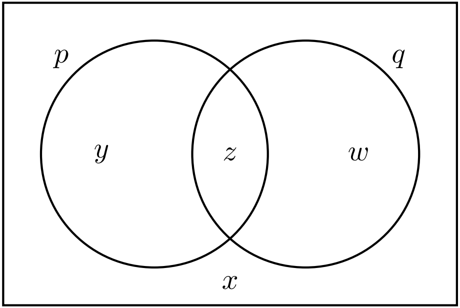
1. $p \supset q \not\models_P p \to q$
2. $p \vee q \not\models_P \neg p \to q$
3. $p \to q \not\models_P \neg q \to \neg p$

We have already seen a counterexample for (1). For (2), consider the following:



Since we have just one premise, we need to show that $P(p \vee q) > P(q|\neg p)$. Indeed, $0.991 > 0.1$.

Some Valid Inferences

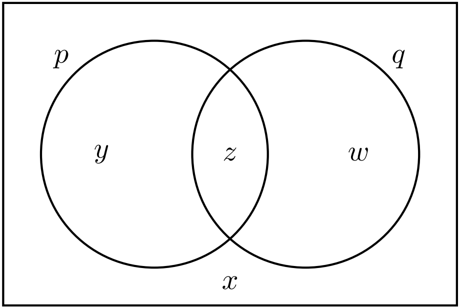


$$p \to q \models_P p \supset q$$

$$P(p \supset q) = x + z + w \geq \frac{z}{y+z} = P(p \to q)$$

Note that $(1 - (x + z + w)) \geq \frac{y}{y+z}$ (because $y + z \leq 1$) and $\frac{y}{y+z} = 1 - \frac{z}{y+z}$.

Some Valid Inferences



$$p, p \to q \models_P q$$

$U_P(q) = x + y$, $U_P(p) = x + w$ and $U_P(p \to q) = \frac{y}{y+z}$

Thus $U_P(q) \leq U_P(p) + U_P(p \to q)$

Note in fact that $x + y \leq x + w + \frac{y}{y+z}$, since $y \leq \frac{y}{y+z}$

Exercise

Show that **transitivity** is not valid:

$$p \to q, q \to r \not\models_P p \to r$$

Show that **cut** is valid:

$$p \to q, (p \wedge q) \to r \models_P p \to r$$

Adams' logic

If we do not allow for embedding of conditionals, Adams (1975) showed that the resulting logic is equivalent to the system $\mathbf{P}$ that we encountered when studying counterfactuals and non-monotonic logics!

Stalnaker (1970) hypothesis: for every conditional $\phi \to \psi$, where ϕ and ψ are not necessarily $\to$-free, $P(\phi \to \psi) = P(\psi|\phi)$, with $P(\phi) \geq 0$.

Unfortunately, Lewis showed that this leads to a triviality result, as the probability of the conditional collapses into the unconditional probability of the consequent.

This result hinges (also) on the so-called import-export principle:

$$P(\phi \to (\psi \to \chi)) = P((\phi \wedge \psi) \to \chi)$$

Outline

An Example

Imagine rolling a die.

p : 'the die landed on a 2'
q : 'the die landed on an even number'
r : 'the die landed on a number < 4'.

(5) a. If the die landed on a number less than 4, it landed on 2.
b. $P(r \to p) = P(p|r) = \frac{1}{3}$

If we allow import-export.

(6) a. If the die landed on an even number, then if it landed on a number less than 4, it landed a 2.
b. $P(q \to (r \to p)) = P((q \wedge r) \to p) = P(p|(q \wedge r)) = 1$

However, $P(q \to (r \to p)) = P((r \to p)|q) \leq \frac{P(r \to p)}{P(q)} = \frac{2}{3}$

Lewis Triviality Result

1. $P(\phi \to \psi) = P(\psi|\phi)$ Adams Thesis
2. $P(\phi \to (\psi \to \chi)) = P((\phi \wedge \psi) \to \chi)$ Import-Export
3. $\phi \to \psi$ is a proposition Stalnaker Thesis

Triviality Result

1. $P(\phi \to (\psi \to \chi)) = P((\phi \wedge \psi) \to \chi)$ Import-Export
2. $P((\psi \to \chi)|\phi) = P(\chi|(\phi \wedge \psi))$ 1, Adams Thesis
3. $P(\phi) = P(\phi \mid \psi) \cdot P(\psi) + P(\phi \mid \neg\psi) \cdot P(\neg\psi)$ Law of Total Probability
4. $P(\phi \to \psi) = P(\phi \to \psi \mid \psi) \cdot P(\psi) + P(\phi \to \psi \mid \neg\psi) \cdot P(\neg\psi)$ As above
5. $P(\phi \to \psi) = P(\psi \mid \phi \wedge \psi) \cdot P(\psi) + P(\psi \mid \phi \wedge \neg\psi) \cdot P(\neg\psi)$ by 2, 4
6. $P(\psi \mid \phi) = P(\psi \mid \phi \wedge \psi) \cdot P(\psi) + P(\psi \mid \phi \wedge \neg\psi) \cdot P(\neg\psi)$ Adams thesis
7. $P(\psi \mid \phi \wedge \psi) = 1$ fact
8. $P(\psi \mid \phi \wedge \neg\psi) = 0$ fact
9. $P(\psi \mid \phi) = P(\psi)$

What to give up?

What to give up

- Import-export (Bradley 2000)
- $\phi \to \psi$ is not a proposition (Adams)
- Adams's thesis (alternative analysis of indicative conditionals)
- Law of probabilities

Problems of Probabilistic Approach

- Some inferences predicted not valid, but they seem to be. How to account for this? Or-to-If, Contraposition, Transitivity
- Some embedded conditionals are meaningful:

  (7) If (the cup broke, if it was dropped), it was fragile.

  (8) It is not the case that if I push this button, the light goes on.

- Intuitively, indicative and counterfactual conditionals are much alike. How to account for the similarity?

  Indicatives truth-conditional after all? — Grice

  Counterfactuals also probabilistically?[2] — Adams, Skyrms

[2] As mentioned in class, we cannot simply take the plain conditional probability of a *counter*factual. Adams proposes–though this view has not gained widespread support–that the 'conditional probability' assigned at the time of utterance to a counterfactual statement like 'If she had taken the medication, she would have recovered' should correspond to the probability you assign to the corresponding indicative statement 'If she takes the medication, she will recover.' on a prior occasion (Adams 1975, ch. IV)