Vagueness III

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Readings

Optional:

- Cobreros, Pablo & Tranchini, Luca (2019). *Supervaluationism, Subvaluationism and the Sorites Paradox*. In Sergi Oms & Elia Zardini (eds.), *The Sorites Paradox*. New York, NY: Cambridge University Press. pp. 38-62.
- Williamson, Timothy (2002). Vagueness. Routledge. (Chapters 7-8)
- van Rooij, Robert (2011). *Vagueness and linguistics*. Vagueness: A guide. Dordrecht: Springer Netherlands. pp. 123-170.

Outline

Supervaluationism and Higher-order Vagueness

Suppose that we add the determinacy operator Δ, discussed before in the context of higher-order vagueness.

Δ can be thought as a necessity modal operator, where each presification acts as a possible world.

Δp will be true in a precisification v' of a three-valued valuation v iff p holds in all precisifications of v.

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Δp will be true in a precisification v' of a three-valued valuation v iff p holds in all precisifications of v.

This leads to modalized versions of supervaluationism where we work with a set of valuations as a set of possible worlds.

We work with a simplified version and take the accessibility relation to be universal.

Modalized Supervaluationism

Given a non-empty set of classical valuations V, we define the pointed satisfaction relation of ϕ for an element $v \in V$, $V, v \models \phi$, as follows.

$$\begin{array}{lll} V, v \models p & \text{iff} & v(p) = 1 \\ V, v \models \phi \wedge \psi & \text{iff} & V, v \models \phi \,\&\, V, v \models \psi \\ \ldots & \ldots & \ldots \\ V, v \models \Delta\phi & \text{iff} & \forall v' \in V : V, v' \models \phi \end{array}$$

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Given a non-empty set of valuations V, a formula ϕ is supertrue iff $V, v \models \phi$ for all $v \in V$. We write $V \models^{!1} \phi$

We use here the *global* version of logical consequence:

$\Gamma \models \phi$ iff for all non-empty set of valuations V if $V \models^{!1} \gamma$ for all $\gamma \in \Gamma$, then $V \models^{!1} \phi$.

Some examples

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Take $V = \{v_1, v_2\}$ with $v_1(p) = 1$ and $v_2(p) = 0$. Now, we have that $V, v_1 \models^{!1} p$ and $V, v_2 \not\models^{!1} p$. But $V, v_1 \not\models^{!1} \Delta p$ and $V, v_2 \not\models^{!1} \Delta p$. Hence, $V, v_1 \not\models^{!1} p \rightarrow \Delta p$ and thus $V \not\models^{!1} p \rightarrow \Delta p$, which suffices to show that $\not\models^{!1} p \rightarrow \Delta p$.

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What does this tell us?

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In general, we can show that for any formula ϕ: $\phi \models \Delta\phi$

$\not\models p \to \Delta p$

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What does this tell us? The deduction theorem fails!
$\Gamma \models \phi \not\Rightarrow \models \Gamma \to \phi$

Higher-order Vagueness

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What to do?

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What to do?

Should Δ obey a weaker logic? Does this solve all the problems of higher-order vagueness? (more on this in the assignment!)

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How do these two theories deal with borderline cases of vagueness?

Determination

Supervaluationism: vagueness as *underdetermination*. Borderline cases are neither supertrue nor superfalse.

Subvaluationism: vagueness as *overdetermination*. Borderline cases are both subtrue and subfalse.

Subtrue and Subfalse

A formula is *sub-true* when it is true in some of its precisification.

Given a three-valued valuation v, a formula ϕ is subtrue with respect to v iff $v'(p) = 1$ for some v' s.t. $v \leq v'$. We write $v \models_{!1} \phi$

A formula is *sub-false* when it is false in some its precisification.

Given a three-valued valuation v, a formula ϕ is sub-false with respect to v iff $v'(\phi) = 0$ for some v' s.t. $v \leq v'$. We write $v \models_{!0} \phi$.

Logical Consequence

Logical consequence is defined as preservation of sub-truth:

$\Gamma \models \phi$ iff for all three-valued valuations v if $v \models_{!1} \gamma$ for all $\gamma \in \Gamma$, then $v \models_{!1} \phi$.

Assessing the Situation

We know the supervaluationist answer: not all conditionals are supertrue, but this does not commit us to take such conditional as superfalse.

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What is the situation in case of subvaluationism?

Modus ponens fails in subvaluationism. Even if the premises are all subtrue, we can make the conclusion not subtrue, as modus ponens does not preserve subtruth.

A dual picture

Consider the inductive form of the argument

(A) $\forall n(\phi(n) \rightarrow \phi(n+1))$

and its negation

(¬A) $\exists n(\phi(n) \wedge \neg\phi(n+1))$

For a supervaluationist, the superfalsity of (A) does not lead to a superfalse instance. Likewise, the supertruth of (¬A) does not lead to a supertrue instance.

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For a supervaluationist, the superfalsity of (A) does not lead to a superfalse instance. Likewise, the supertruth of ($\neg$A) does not lead to a supertrue instance.

For a subvaluationist, the subtruth of all the instances of (A) does not lead to the subtruth of (A). Likewise, the subfalsity of all the instances of ($\neg$A) does not lead to the subfalsity of ($\neg$A).

Supervaluations and classical logic

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But adding the Δ operator invalidates certain classical meta-inferences (e.g., the deduction theorem).

Supervaluations and truth-functionality

A further problem: supervaluationist (and subvaluationist) theories are not truth-functional and thus not compositional.

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This is not satisfied in this setting (e.g., the supertruth of $\neg p$ is not determined by the supertruth of p)

$$f_{\neg}(t) = \begin{cases} \text{supertrue} & \text{if } t = \text{not-supertrue} \\ \text{not-supertrue} & \text{if } t = \text{supertrue} \end{cases}$$

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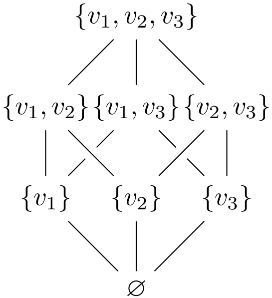
What about disjunction or conjunction?

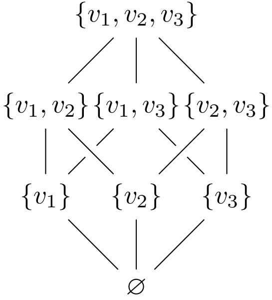
An algebraic perspective

Given a set of valuations V, the power set of V forms a Boolean algebra.

The elements of the generated structure can be thought as the 'values' of formulas.

Take $V = \{v_1, v_2, v_3\}$ with $v_1(p) = v_2(q) = 1$ and 0 otherwise. Then p corresponds to $\{v_1\}$, $p \vee q$ to $\{v_1, v_2\}$





'Functionality' of formulas is preserved in the sense:

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\begin{array}{lcl}
f(p) & = & \{v \in V : v(p) = 1\} \\
f(\neg\phi) & = & V \backslash f(\phi) \\
f(\phi \vee \psi) & = & f(\phi) \cup f(\psi) \\
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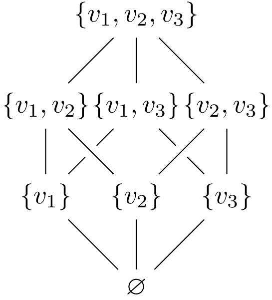

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$$

ϕ is supertrue in V iff $f(\phi) = V$
ϕ is superfalse iff $f(\phi) = \varnothing$

Supervaluations and degrees

$$
\begin{tikzcd}
& \{v_1, v_2, v_3\} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\{v_1, v_2\} \arrow[d, dash] \arrow[dr, dash] & \{v_1, v_3\} \arrow[dl, dash] \arrow[dr, dash] & \{v_2, v_3\} \arrow[dl, dash] \arrow[d, dash] \\
\{v_1\} \arrow[dr, dash] & \{v_2\} \arrow[d, dash] & \{v_3\} \arrow[dl, dash] \\
& \varnothing &
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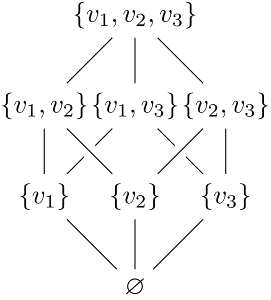
Field (2008) argues that this Boolean representation highlights that supervaluationism is in effect a degree-theoretic approach.

Supervaluations and degrees



Field (2008) argues that this Boolean representation highlights that supervaluationism is in effect a degree-theoretic approach.

However, take again $V = \{v_1, v_2, v_3\}$ with $v_1(p) = v_2(q) = 1$ and 0 otherwise. Then $f(p) = \{v_1\}$ and $f(\neg p) = \{v_2, v_3\}$. But for supervaluationism p is not 'less true' than $\neg p$.

Outline

The epistemic solution

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Vagueness doesn’t entail non-existence of cutoffs, but rather the unlocatability of such cutoffs due to the nature of our knowledge.

Inexact knowledge

One of the major proponents of the epistemic theory is Timothy Williamson (*Vagueness*, 1994).

Our knowledge of the application of a vague term is **inexact**.

In contexts where knowledge is imprecise, a **margin of error** indicates the range within which knowledge can be considered reliable.

Fixed margin models

A fixed margin model is a Kripke model $M = \langle W, R, V \rangle$, with R determined by a metric/function d on W and an error parameter α s.t. $R(x, y)$ iff $d(x, y) \leq \alpha$.

In particular, we have that for all $x, y, z \in W$:

- $d(x, y) = 0$ iff $x = y$
- $d(x, y) = d(y, x)$
- $d(x, z) \leq d(x, y) + d(y, z)$

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What constraints on R does this metric impose? Reflexivity and symmetry.

Knowledge Operator

$M, x \models \Box\phi$ iff $\forall y$ s.t. $R(x, y)$ (i.e., $d(x, y) \leq \alpha$), $M, y \models \phi$

$\Box\phi$ is true at x when 'I know that ϕ holds at every world included within the margin α from x'.

[Égré and Bonnay 2010]

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In what follows, we consider a simplified model by taking $W = \mathrm{N}$ and $\alpha = 1$.

[Égré and Bonnay 2010]

An example

Take $W = \mathrm{N}$ and $\alpha = 1$, and consider the model below.

$$\begin{tikzcd}[row sep=tiny]
0 \arrow[loop above] \arrow[r, leftrightarrow] & 1 \arrow[loop above] \arrow[r, leftrightarrow] & 2 \arrow[loop above] \arrow[r, leftrightarrow] & 3 \arrow[loop above] \arrow[r, leftrightarrow] & 4 \arrow[loop above] \arrow[r, leftrightarrow] & 5 \arrow[loop above] \arrow[r] & \cdots \\
p & p & p & p & \neg p & \neg p &
\end{tikzcd}$$

Setting $\alpha = 1$ for R means that I cannot discriminate any heap ≤ 1 distant (i.e., any heap from itself, any two adjacent heaps) but I can discriminate between any two non-adjacent heaps.

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\underset{p}{0} \arrow[loop above] \arrow[r, leftrightarrow] & \underset{p}{1} \arrow[loop above] \arrow[r, leftrightarrow] & \underset{p}{2} \arrow[loop above] \arrow[r, leftrightarrow] & \underset{p}{3} \arrow[loop above] \arrow[r, leftrightarrow] & \underset{\neg p}{4} \arrow[loop above] \arrow[r, leftrightarrow] & \underset{\neg p}{5} \arrow[loop above] \arrow[r] & \cdots
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Let $n \models \Box p$ be by considering a collection of 10^n grains of sand, 'I *know* that it does not make a heap'.

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$$\begin{array}{ccccccccccccc} 0 & \longleftrightarrow & 1 & \longleftrightarrow & 2 & \longleftrightarrow & 3 & \longleftrightarrow & 4 & \longleftrightarrow & 5 & \longrightarrow & \cdots \\ p & & p & & p & & p & & \neg p & & \neg p & & \end{array}$$

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Let $n \models \Box p$ be by considering a collection of 10^n grains of sand, 'I *know* that it does not make a heap'.

$2 \models \Box p$ and $5 \models \Box\neg p$

But $3 \models \neg\Box p \wedge \neg\Box\neg p$ and $4 \models \neg\Box p \wedge \neg\Box\neg p$

Assessing the Epistemic Response

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If there is a cut-off, **why cannot we know it**?

Being *tall* is vague, and we do not know what counts as tall. Being *the largest twin number* is **precise**. Yet we do not know the latter.

Assessing the Epistemic Response

The height of John is 2 meters. We know that ‘John is tall’, and this knowledge is **by no means inexact**.

Why then does inexact knowledge apply only to borderline cases? And what counts as a borderline case?

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The height of John is 2 meters. We know that ‘John is tall’, and this knowledge is **by no means inexact**.

Why then does inexact knowledge apply only to borderline cases? And what counts as a borderline case?

If knowledge is inexact within a margin of error, do we *know* this margin of error, or is it also inexact?