Introduction and Vagueness

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Philosophical Logic 2024
29 October 2024

Course Team

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Please note that the location of the classes changes quite frequently throughout the course.

`https://datanose.nl/#course[128585]`

Course Structure

There will be 2 lectures + 1 tutorial each week.

We will cover foundational topics in philosophical logic, introducing you to different non-classical logic systems developed to address specific philosophical problems.

As support material, there will be the slides, and set of lecture notes, and pointers to the literature for further recommended reading.

The tutorials are practice exercises sessions over the material presented in class.

Assessment

50% assignments + 50% final exam

The assignments will cover both **philosophical** exercises (writing/reflection) and **technical** exercises.

New assignments will be released every Thursday at 12 noon and will be due the following Wednesday at 9pm (5 assignments in total).

You are allowed to discuss the exercises with your classmates. But the writing must be done **individually**. No need to disclose the names of classmates with whom you discussed.

The exam will include technical exercises as well as one philosophical question.

Final Exam

The current final exam is scheduled for Wednesday 18 December from 9 to 11. This overlaps with the exam for Proof Theory.

Some options:

- Wednesday 18: 17-19
- Thursday 19: 9-11
- Thursday 19: 13-15
- Thursday 19: 17-19

A survey will be available on Canvas.

Provisional Schedule

Week 1 and Week 2: Intro & Vagueness (paradoxes, many-valued logics, fuzzy logics, supervaluations, subvaluations, epistemic analyses, contextualist solutions)

Week 3 and Week 4: Truth (general theories, Tarski hierarchy, Kripke theory of truth, self-reference paradoxes, truthmakers)

Week 5 and Week 6: Conditionals (possible-world analyses, premise semantics, conditionals and modality, non-monotonic logics, conditionals and probabilities)

Week 7: Future contingents, and practice exam

Course Feedback

The university will organize a detailed evaluation after the course ends, which is helpful for future improvements.

However, if you have feedback during the course, please don't hesitate to contact us. We're always open to suggestions and criticism to improve the course in real-time.

To share your anonymous feedback, whether positive or critical, you can also use the following Google form:

`https://forms.gle/mnsww6H7se3rn4qx9`

Plan

1. Philosophical Logic
2. Vagueness
3. Many-valued Logics
4. Three-valued Logics

Readings

Required:

- Lecture notes: ch. 1; ch. 2.1-2.2; ch. 3.1-3.2

Optional:

- An Introduction to Non-Classical Logic (Priest): ch. 1.1-1.3; ch. 7.1-7.3
- Logic for Philosophy (Sider): ch. 3.4.1-3.4.3

Outline

Philosophical Logic vs Philosophy of Logic

What is the role played by ‘logic’ in philosophical logic?

Logic: Formal system to regiment *reasoning* by means of a formal language (e.g., rules of inference, valid inference, completeness, consistency, axiomatization, . . .).

Philosophy of Logic: the philosophical study of ‘logic’ and its fundamental concepts (e.g., the nature of *entailment*, the *truth* of a logical statement, . . .)

Philosophical Logic: the application of logic(s) to philosophical problems (e.g., knowledge and epistemic logics, conditionals, vagueness, . . .)

The domains of inquiry of philosophy of logic and philosophical logic are in many respects interconnected.

Philosophical logic and classical logic

One way to conceive philosophical logic is the study of logics which are not-classical (intuitionistic logic, relevance logic, $\ldots$).

The law of excluded middle is valid in classical logic. But it is not valid in intuitionistic logic.

$$p \vee \neg p$$

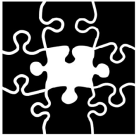
Intuitionistic logic rejects non-constructive proofs and links 'truth' with 'verifiability'.

Example: There exist irrational x, y such that x^y is rational.

[Burgess 2009; Prawitz 2005]

Philosophical logic and paradoxes

Many philosophically interesting problems and logics emerge as solutions seeking to solve a particular **paradox** or **puzzle**.



Paradoxes will often be our starting point, leading to the study of logical theories which can account for them.

Omnipotence and the paradox of the stone

Could God create a stone so heavy that even God could not lift it?

1. God is omnipotent (i.e., God can do anything)

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3. If God *can not* create a stone that God can not lift, then God is not omnipotent.
4. God is not omnipotent.

The Liar Paradox

1. This sentence is false.

The Liar Paradox

1. This sentence is false.
2. If this sentence is false, then it is true.

The Liar Paradox

1. This sentence is false.
2. If this sentence is false, then it is true.
3. If this sentence is true, then it is false.

Epistemology and self-refutation

Relativism: There is no absolute truth.

Scepticism: Nothing can be known.

Do relativists take as **true** that there is no absolute **truth**?

Do sceptics **know** that nothing can be **known**?

Russell's Paradox

Let A be the set of all sets that are not members of themselves.

If A is not a member of itself, then by definition it is a member of itself

If A is a member of itself, then it is not a member of A, since it is the set of all sets that are not members of themselves.

Berry's Paradox

The smallest positive integer not definable in under 200 letters.

Let X be the set of all positive integers definable in under 200 letters. Sentences under 200 letters are finite and thus X is finite.

There are infinitely many positive integers, so there exist positive integers which do not belong to X.

The sentence above contains less than 200 letters and defines a positive integer.

By definition, this integer is not definable in under 200 letters.

The role of paradoxes

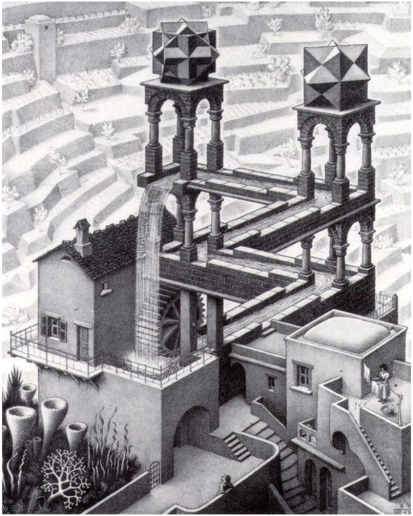
All these paradoxes rest on a number of (controversial) assumptions.

We will explore how to identify the relevant assumptions associated with a paradox and determine its *structure*.

By doing so, we will be able to show that different paradoxes exhibit similar structures and thus call for a unitary solution.

Paradoxes

But what is a paradox?



Waterfall - M. C. Escher

Defining a paradox

A paradox is an argument with **assumptions** which appear to be true and steps which appear to be valid, which nevertheless ends in a conclusion which is false.

What are examples of logical assumptions? And non-logical ones?

An example: the liar paradox

A: A is false.

- A truth predicate T to the language
- We take $\ulcorner A \urcorner$ as the name for a sentence A
- T-schema: $T(\ulcorner A \urcorner) \leftrightarrow A$

An example: the liar paradox

1. $T(\ulcorner A \urcorner) \vee \neg T(\ulcorner A \urcorner)$	LEM
2. $T(\ulcorner A \urcorner)$	Hyp
3. A	T-schema
4. $\neg T(\ulcorner A \urcorner)$	Def.
5. $T(\ulcorner A \urcorner) \wedge \neg T(\ulcorner A \urcorner)$	2, 5 $\wedge$-I
6. $\neg T(\ulcorner A \urcorner)$	Hyp
7. A	Def.
8. $T(\ulcorner A \urcorner)$	T-schema
9. $T(\ulcorner A \urcorner) \wedge \neg T(\ulcorner A \urcorner)$	6, 8 $\wedge$-I
10. $T(\ulcorner A \urcorner) \wedge \neg T(\ulcorner A \urcorner)$	Reasoning by cases

What to give up?

Extra-logical assumptions: truth predicate, T-schema (which direction?)

Logical assumptions: LEM, explosion, $\wedge$-Introduction, reasoning by cases, …

The road to philosophical logic

To challenge each of these assumptions, one must both philosophically understand the motivations for questioning them and formally establish a well-structured logical system.

Philosophical logic!

Outline

Tall, bald and red

Informally, vagueness is a property of words, phrases, or concepts that **lack clear boundaries** in their meaning.

Tall: When is someone considered tall?
Bald: How many hairs can someone have before they are considered bald?
Red: At what point the color is not red?

Distinguishing Vagueness

Are vague words ambiguous?

*There is a duck by the **bank**.*

No. *Bank* has two distinct meanings (a financial institution or riverbank), so it's ambiguous, not vague.

Are vague words context-dependent?

Not always. *Tall* is both vague and context-dependent. *Bush* is vague but not context-dependent.

Are vague words under-specified?

Yes, although under-specification is a broader category.

[Van Rooij 2011]

Criteria for Vagueness

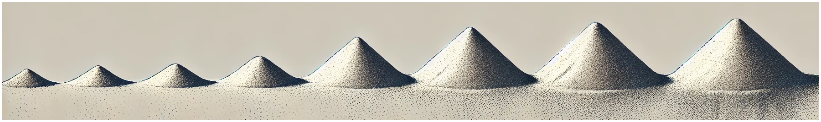
- Lack of sharp boundaries
- Presence of borderline cases
- Tolerant to small differences along a relevant dimension
- Can lead to the sorites paradox

The Sorites paradox

1 grain of sand does not make a heap.
If 1 grain don't make a heap, then 2 grains don't.
If 2 grains don't make a heap, then 3 grains don't.
...
If 1M-1 grains don't make a heap, then 1M grains don't.

---

1 million grains don't make a heap.



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$$\begin{array}{l} \phi(1) \\ \phi(1) \to \phi(2) \\ \phi(2) \to \phi(3) \\ \dots \\ \phi(1M-1) \to \phi(1M) \\ \hline \phi(1M) \end{array}$$

$$\begin{array}{l} \phi(1) \\ \forall n(\phi(n-1) \to \phi(n)) \\ \hline \forall n(\phi(n)) \end{array}$$

The Sorites paradox

$$\begin{array}{l}
\phi(1) \\
\phi(1) \rightarrow \phi(2) \\
\phi(2) \rightarrow \phi(3) \\
\ldots \\
\phi(1M-1) \rightarrow \phi(1M) \\
\hline
\phi(1M)
\end{array}$$

We have a number of plausible assumptions and 1 inference rule: **modus ponens.** What to do?

Reject an **assumption**? $\exists n(\phi(n-1) \wedge \neg\phi(n))$

Reject **modus ponens**?

$\ldots$

Outline

Classical Logic

Logic studies *valid* reasoning.

Logic is a formal language with a deductive system and/or a semantics.

Formal language (object language): set of well-formed strings over a finite alphabet

$$\phi ::= p|\bot|\top|\neg\phi|\phi \vee \phi|\phi \wedge \phi|\phi \to \phi$$

An argument is *derivable* if there is deductive procedure to derive its conclusion from its premises. An argument is *valid* if whenever its premises are all true, its conclusion is false.

[Priest 2008, ch. 1]

Semantics

We can specify the semantics of our language with a *valuation* function.

Let P be set of propositional letters in our language. A valuation v is a function $v : P \rightarrow \{0, 1\}$ extended recursively over the formulas in the language:

$$
\begin{aligned}
&v(\top) = 1 \\
&v(\bot) = 0 \\
&v(\neg\phi) = 1 - v(\phi) \\
&v(\phi \wedge \psi) = v(\phi) * v(\psi) \\
&v(\phi \vee \psi) = \ldots
\end{aligned}
$$

The valuation function is often conveniently represented by means of truth-tables.

Some important notions

A formula ϕ is satisfiable by a valuation v iff $v(\phi) = 1$

A formula ϕ is valid iff $v(\phi) = 1$ for any valuation v.

Given a set of formulas Γ and a formula ϕ, we say that Γ entails ϕ and we write $\Gamma \models \phi$ iff all valuations v that make all $\gamma \in \Gamma$ true make ϕ true.

Based on this, what changes to the logic can be made to depart from the classical picture?

Bivalence and Vagueness

Bivalence: Every sentence is true (1) or false (0).

Proposal: in borderline cases, conditionals of the form $\phi(n-1) \rightarrow \phi(n)$ are neither true nor false.

A **third truth-value** besides 1 and 0.

We then need to specify the truth tables of our connectives by a valuation function over the formulas in our language with respect to this additional truth value.

To define everything accordingly, we will start by considering the more general **many-valued logics** and then return to the Sorites paradox.

Many-Valued Logics

Given a language $\mathcal{L}$, the general make-up of a many-valued logic will formed by

1. A finite non-empty set of truth values T
2. A set $T^+ \subseteq T$ of designated truth values (for validity)
3. For each n-place connective, a truth value function $v : T^n \to T$. If $n = 0$, $v(\cdot) \in T$

These three elements form the **logical matrix** of $\mathcal{L}$.

How to represent classical logic? Let $\mathcal{L}_{CL}$ be the language with connectives $\bot$, $\neg$, $\wedge$, $\vee$, $\to$ as usual.

1. $T = \{1, 0\}$
2. $T^+ = \{1\}$
3. The usual truth tables. $v(\bot) = 0$

Validity and Entailment

A formula ϕ is satisfiable by a valuation v iff $v(\phi) \in T^+$

A formula ϕ is valid iff $v(\phi) \in T^+$ for all valuations v.

Given a set of formulas Γ and a formula ϕ, we say that Γ entails ϕ and we write $\Gamma \models \phi$ iff for any valuation v s.t. $v(\gamma) \in T^+$ for all $\gamma \in \Gamma$, then $v(\phi) \in T^+$.

Can you think of alternative ways to define entailment?

Outline

Three-valued logics

Three-valued logics are many-valued logics with an additional truth value besides 1 and 0. $T = \{1, 0, i\}$

Two modelling choices:

1. Truth value function for **connectives**.
2. Entailment and validity depending on the designated value(s) in T^+.

For the moment, we take $T^+ = \{1\}$ and just look at the truth value function.

Truth-tables

$\wedge$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	0	$?$	0

$\vee$	1	i	0
1	1	$?$	1
i	$?$	$?$	$?$
0	1	$?$	0

$\rightarrow$	1	i	0
1	1	$?$	0
i	$?$	$?$	$?$
0	1	$?$	1

$\neg$	
1	1
i	$?$
0	0

Take $\wedge$. How many truth value functions can you define?

Some natural constraints:

Idempotence: $p \wedge p \equiv p$

Symmetry: $p \wedge q \equiv q \wedge p$

How many truth value functions for $\wedge$?

Truth-tables (Strong Kleene K_3^s)

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	1	i	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

$p \rightarrow q \equiv \neg p \vee q$

Role of i:

What is $p \leftrightarrow q := p \rightarrow q \wedge q \rightarrow p$?

Truth-tables (Weak Kleene K_3^w)

$\wedge$	1	i	0
1	1	i	0
i	i	i	i
0	0	i	0

$\vee$	1	i	0
1	1	i	1
i	i	i	i
0	1	i	0

$\rightarrow$	1	i	0
1	1	i	0
i	i	i	i
0	1	i	1

$\neg$	
1	0
i	i
0	1

$p \rightarrow q \equiv \neg p \vee q$

Role of i:

What is $p \leftrightarrow q := p \rightarrow q \wedge q \rightarrow p$?

Some Facts

Both K_3^s and K_3^w have no tautologies. Why?

$v_\vee(i,i) = v_\wedge(i,i) = v_\to(i,i) = v_\neg(i) = i$

Still, the consequence relation is not trivial. Can you think of some cases where K_3^s and K_3^w diverge?

$p \models_{K_3^s} p \vee q$ $\qquad$ $p \not\models_{K_3^w} p \vee q$

Another look at the valuation function

We can view the undefined value i as $\frac{1}{2}$

$$0 < \frac{1}{2} < 1$$

We can then define the truth value function for K_3^s using:

$$v(p \wedge q) = \mathsf{min}(v(p), v(q))$$
$$v(p \vee q) = \mathsf{max}(v(p), v(q))$$
$$v(\neg p) = 1 - v(p)$$
$$v(p \to q) = \mathsf{max}((1 - v(p)), v(q))$$

This characterization of the evaluation function will be helpful when we will generalize the logic to (infinitely-)many values. And it also gives another perspective on the role of the truth value function.

Łukasiewicz three-valued logic Ł3

$\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

$\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$\to$	1	i	0
1	1	i	0
i	1	$\mathbf{1}$	i
0	1	1	1

$\neg$	
1	0
i	i
0	1

$p \to q \not\equiv \neg p \vee q$

There are no tautologies in K_3^s and K_3^w. What about Ł3?

What is $p \leftrightarrow q := p \to q \wedge q \to p$?

A Basic Fact

Let $\mathcal{T}_{CL}$ and $\mathcal{T}_{Ł_3}$ be the set of classical and Ł3 tautologies.

It holds that:

$$\mathcal{T}_{Ł_3} \subset \mathcal{T}_{CL}$$

[Call a valuation v' classical if $v'(p) \in \{1, 0\}$ for all propositional letters in the language. Show by induction that for any ϕ, if v' is classical, then $v'_{Ł_3}(\phi) = v'_{CL}(\phi)$]